

# Radiative transfer and diffusion limits for wave field correlations in locally shifted random media\*

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## Abstract

The aim of this paper is to develop a mathematical framework for opto-elastography. In opto-elastography, a mechanical perturbation of the medium produces a decorrelation of optical speckle patterns due to the displacements of optical scatterers. To model this, we consider two optically random media, with the second medium obtained by shifting the first medium in some local region. We derive the radiative transfer equation for the cross-correlation of the wave fields in the media. Then we derive its diffusion approximation. In both the radiative transfer and the diffusion regimes, we relate the correlation of speckle patterns to the solutions of the radiative transfer and the diffusion equations. We present numerical simulations based on our model which are in agreement with recent experimental measurements.

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**Key words.** opto-elastography, radiative transfer equation, cross-correlation techniques

## 1 Introduction

When a strongly scattering medium is illuminated by a coherent laser, the transmitted light forms a speckle pattern whose statistical properties depends on the statistical properties of the medium. When the optical scatterers are displaced in some region, the speckle pattern changes and its correlation with the original speckle pattern depends on the amplitude of the displacement and on the local optical properties (scattering and absorption) of the region affected by the displacement.

Focused ultrasound introduces such displacement of scatterers through two different mechanisms. First, ultrasound focusing generates oscillating compressive strain in the focal region; the oscillation of the optical scatterers is in the MHz frequency range. Second, high-intensity

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focused ultrasound may also generate low frequency (of kHz range) strain in elastic media, which in turn generates shear wave propagating in the medium. Both modifications of scatterers can cause decorrelation of speckle patterns. However, when speckle patterns are recorded with an exposure time sufficiently long with respect to the compressive oscillation while remains short with respect to the shear motion, the effect of the high frequency compressive motion averages out and the decorrelation in speckle patterns is dominated by the second mechanism. Based on this idea, transient opto-elastography experiments have been carried out by Bossy *et al.* in [4, 5], where they found qualitative relations between the decorrelation of speckle patterns and optical absorption as well as mechanical properties (such as Young's modulus) of soft biological tissues.

The main objective of this paper is to provide an analytical model that relates the decorrelation of speckle patterns to the displacement of the scatterers. Such a model hence helps us to understand the aforementioned "acousto-elasto-optic" phenomenon. To this end, we start with the Helmholtz equation with the refractive index having a highly oscillatory random part. We consider the regime where the correlation length of the random medium is of the same order as the wavelength and both are smaller than the typical propagation distance. We denote by  $\varepsilon$  the ratio between the correlation length of the medium and the propagation distance and assume therefore that  $\varepsilon \ll 1$ . We also assume that the relative amplitude of the random fluctuations is weak, of order  $\sqrt{\varepsilon}$ . This is known to be a scaling regime where the random medium interacts with the propagating high-frequency waves [11]. We consider two random media, with the second medium obtained by shifting the first medium in some local region.

The correlation of wave fields is well described by the Wigner distributions [9]. Following the techniques of [11], we formally derive the radiative transfer equations (RTEs) for such cross-correlations of wave fields in two random media in the limit as  $\varepsilon$  goes to zero. Radiative transfer limits for waves in two different random media have been considered, e.g. in [1, 2], but the case of two media related by a local shift considered in this paper is new. The salient effects of this local shift of random media are: it introduces a phase modulation to the scattering kernel of the RTE in the region affected by the shift; further, when the amplitude of the shift is large, say much larger than the wavelength, by non-stationary phase the RTE for the cross-correlation function is intrinsically absorbing in this region. Next, following the techniques of [8, 3, 7], we derive the diffusion approximation of the RTE in the regime when the mean free path is small; this simplification is useful for numerical simulations. In the special case of large shift, the cross correlation vanishes in the region affected by the shift. We shall see that this accounts for the loss of correlation of the speckle patterns.

The paper is organized as follows. In the next section, we present the model of two random media considered in this paper; they are related by a local shift. In Section 3, we derive the RTEs for the Wigner distributions of the wave fields in the aforementioned two random media, in the limit when  $\varepsilon$  goes to zero; we also present the formula for speckle pattern correlation in terms of the solutions of these RTEs. In Section 4, we derive the diffusion approximation of the RTEs in the limit when the mean free path goes to zero, and again we derive the formula for the speckle pattern correlation in the diffusion regime. In the special case of large shift, using the derived diffusion approximation and correlation formula, we run numerical simulations which confirm that the diffusion equation model is able to capture the loss of correlation of the speckle patterns.

## 2 Wave equation and heterogeneous media

In the microscopic scale, light propagation is described by the Maxwell equations. In the case when scalar approximation is valid, these reduce to the following Helmholtz equation for the electric field  $u$ ,

$$\Delta u(\mathbf{x}) + k_0^2 n^2(\mathbf{x}) u(\mathbf{x}) = 0, \quad (1)$$

where  $k_0$  is the wave number of the light in vacuum and the spatially varying refractive index  $n^2(\mathbf{x})$  models the heterogeneous medium.

The following model for  $n^2$  is adopted:

$$n^2(\mathbf{x}) := n_0^2 \left( 1 + 2\sigma V\left(\frac{\mathbf{x}}{l}\right) \right), \quad (2)$$

where  $V(\mathbf{x})$  is a real valued mean-zero random process. Therefore, the heterogeneous medium has mean refractive index  $n_0^2$ , which is assumed to be a constant, and its relative fluctuations are captured by  $2\sigma V(\mathbf{x}/l)$  where the numbers  $\sigma$  and  $l$  model the strength and the correlation length of the fluctuations respectively.

The random process  $V(\mathbf{x})$  is assumed to be stationary, *i.e.*, statistically homogeneous. The two-point correlation function of this process is

$$R(\mathbf{x}) = \mathbb{E}[V(\mathbf{y})V(\mathbf{y} + \mathbf{x})] = \mathbb{E}[V(\mathbf{0})V(\mathbf{x})]. \quad (3)$$

where  $\mathbb{E}$  stands for the expectation with respect to the distribution of the random medium. Throughout this paper, we adopt the convention that the Fourier transform of a function  $f$  on  $\mathbb{R}^d$  is defined by

$$\hat{f}(\boldsymbol{\xi}) := \frac{1}{(2\pi)^d} \int e^{i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x}. \quad (4)$$

Using this notation and the stationarity of  $V$ , one easily verifies that

$$\mathbb{E}[\hat{V}(\mathbf{p})\hat{V}(\mathbf{q})] = \hat{R}(\mathbf{p})\delta(\mathbf{p} + \mathbf{q}), \quad (5)$$

where  $\delta$  is the Dirac distribution. From now on the product  $k_0^2 n_0^2$  is denoted by  $k^2$ .

Let  $L$  denote the typical propagation distance. In the high-frequency regime, the ratio between the wavelength and the propagation distance, denoted by  $\varepsilon := k^{-1}/L$ , is much smaller than one. Let  $\kappa := l/k^{-1}$  be the ratio of the correlation length of the random medium to the wavelength. We consider the so-called weak coupling high-frequency medium which corresponds to  $\kappa = 1$  and  $\sigma = \sqrt{\varepsilon}$ . This is known to be a situation where the heterogeneous medium interacts with the high-frequency waves [11]. Denoting by  $u^\varepsilon(\mathbf{x})$  the scaled function  $u(L\mathbf{x})$ , the Helmholtz equation becomes

$$\frac{\varepsilon^2}{2} \Delta u^\varepsilon(\mathbf{x}) + \frac{1}{2} u^\varepsilon(\mathbf{x}) + \sqrt{\varepsilon} V^\varepsilon(\mathbf{x}) u^\varepsilon(\mathbf{x}) = 0. \quad (6)$$

with  $V^\varepsilon(\mathbf{x}) = V(\mathbf{x}/\varepsilon)$ .

**Two random media.** As mentioned in the Introduction, we are interested in the correlation of two wave fields in two random media. With the application to opto-elastography in mind, we view these two media as configurations of scatterers at two instants of the shear motion. Let  $u_1^\varepsilon$  be the first wave field which solves (6) with  $V_1^\varepsilon(\mathbf{x}) = V(\mathbf{x}/\varepsilon)$ . Let  $u_2^\varepsilon$  be the second wave field which solves (6) with  $V_2^\varepsilon(\mathbf{x})$  given by

$$V_2^\varepsilon(\mathbf{x}) = V_1^\varepsilon(\mathbf{x} + \varepsilon\phi(\mathbf{x})) = V\left(\frac{\mathbf{x}}{\varepsilon} + \phi(\mathbf{x})\right), \quad V_1^\varepsilon(\mathbf{x}) = V\left(\frac{\mathbf{x}}{\varepsilon}\right). \quad (7)$$

Here  $\phi$  is a continuous and compactly supported vector field.  $V_2^\varepsilon$  is obtained from  $V_1^\varepsilon$  by the application of the diffeomorphism  $\mathbf{x} \rightarrow \mathbf{x} + \varepsilon\phi(\mathbf{x})$ . Hence  $\phi$  can be thought as the displacement field that shifts the configurations of optical scatterers. Note that the amplitude of the shift is of order  $\varepsilon$ , which is of the order of the optical wavelength.

**Correlation of waves.** It is well known that the correlation of two scalar wave fields  $u$  and  $v$  is well described by the Wigner transform  $W[u, v]$ , which is defined by

$$W[u, v](\mathbf{x}, \mathbf{k}) := \frac{1}{(2\pi)^d} \int e^{i\mathbf{k} \cdot \mathbf{y}} u(\mathbf{x} - \frac{\varepsilon \mathbf{y}}{2}) \overline{v}(\mathbf{x} + \frac{\varepsilon \mathbf{y}}{2}) d\mathbf{y}, \quad (8)$$

where the bar denotes the complex conjugate. According to (4), the Wigner transform may be thought as the Fourier transform of the two-point correlation function of the fields. In this paper we are interested in

$$W_{jl}^\varepsilon(\mathbf{x}, \mathbf{k}) = W[u_j^\varepsilon, u_l^\varepsilon](\mathbf{x}, \mathbf{k}), \quad (9)$$

where the wave fields  $u_j^\varepsilon$ ,  $j = 1, 2$ , are described earlier and they correspond to two random media  $V_j$  related by (7). We note that

$$\int W_{jl}^\varepsilon(\mathbf{x}, \mathbf{k}) d\mathbf{k} = u_j^\varepsilon(\mathbf{x}) \overline{u_l^\varepsilon}(\mathbf{x}). \quad (10)$$

In particular,  $\int W_{11}^\varepsilon d\mathbf{k}$  is the energy density  $|u_1^\varepsilon|^2$  and  $W_{11}^\varepsilon$  itself corresponds to frequency-specified energy density of the first wave field. The functions  $W_{22}^\varepsilon$  and  $W_{12}^\varepsilon$  can be interpreted similarly.

In the next section, we derive the radiative transfer equation (RTE) for  $W_{jl}^\varepsilon$  in the limit  $\varepsilon \rightarrow 0$ , and we relate the correlation of speckle patterns to these Wigner distributions.

### 3 Radiative transport equation for the Wigner distributions

The goal of this section is to derive the RTE for the correlation of waves. Recall that the two wave fields  $u_j^\varepsilon$ ,  $j = 1, 2$ , solve

$$\frac{\varepsilon^2}{2} \Delta u_j^\varepsilon(\mathbf{x}) + \frac{1}{2} u_j^\varepsilon(\mathbf{x}) + \sqrt{\varepsilon} V_j^\varepsilon(\mathbf{x}) u_j^\varepsilon(\mathbf{x}) = 0, \quad (11)$$

where  $V_2^\varepsilon$  and  $V_1^\varepsilon$  satisfy (7). Recall that  $W_{jl}^\varepsilon$  is defined in (9). For the moment, we assume that  $u_j^\varepsilon$  has prescribed plane wave behavior at infinity to simplify the calculation and derive RTEs for the limits of  $W_{jl}^\varepsilon$ 's. For bounded domains, these RTEs are valid in the interior of the domain, and appropriate boundary conditions should be imposed.

Using (9) and (11), integration by parts and the prescribed plane wave behaviors of  $u_j^\varepsilon$ 's, we find

$$\begin{aligned} i\mathbf{k} \cdot \nabla W_{11}^\varepsilon(\mathbf{x}, \mathbf{k}) &= \frac{1}{\sqrt{\varepsilon}} \int \frac{e^{i\mathbf{k} \cdot \mathbf{y}}}{(2\pi)^d} \left[ V\left(\frac{\mathbf{x}}{\varepsilon} + \frac{\mathbf{y}}{2}\right) - V\left(\frac{\mathbf{x}}{\varepsilon} - \frac{\mathbf{y}}{2}\right) \right] u_1^\varepsilon\left(\mathbf{x} - \frac{\varepsilon \mathbf{y}}{2}\right) \overline{u_1^\varepsilon}\left(\mathbf{x} + \frac{\varepsilon \mathbf{y}}{2}\right) d\mathbf{y}, \\ i\mathbf{k} \cdot \nabla W_{12}^\varepsilon(\mathbf{x}, \mathbf{k}) &= \frac{1}{\sqrt{\varepsilon}} \int \frac{e^{i\mathbf{k} \cdot \mathbf{y}}}{(2\pi)^d} \left[ V\left(\frac{\mathbf{x}}{\varepsilon} + \frac{\mathbf{y}}{2} + \phi\left(\mathbf{x} + \frac{\varepsilon \mathbf{y}}{2}\right)\right) - V\left(\frac{\mathbf{x}}{\varepsilon} - \frac{\mathbf{y}}{2}\right) \right] u_1^\varepsilon\left(\mathbf{x} - \frac{\varepsilon \mathbf{y}}{2}\right) \overline{u_2^\varepsilon}\left(\mathbf{x} + \frac{\varepsilon \mathbf{y}}{2}\right) d\mathbf{y}, \\ i\mathbf{k} \cdot \nabla W_{22}^\varepsilon(\mathbf{x}, \mathbf{k}) &= \frac{1}{\sqrt{\varepsilon}} \int \frac{e^{i\mathbf{k} \cdot \mathbf{y}}}{(2\pi)^d} \left[ V\left(\frac{\mathbf{x}}{\varepsilon} + \frac{\mathbf{y}}{2} + \phi\left(\mathbf{x} + \frac{\varepsilon \mathbf{y}}{2}\right)\right) - V\left(\frac{\mathbf{x}}{\varepsilon} - \frac{\mathbf{y}}{2} + \phi\left(\mathbf{x} - \frac{\varepsilon \mathbf{y}}{2}\right)\right) \right] \\ &\quad \times u_2^\varepsilon\left(\mathbf{x} - \frac{\varepsilon \mathbf{y}}{2}\right) \overline{u_2^\varepsilon}\left(\mathbf{x} + \frac{\varepsilon \mathbf{y}}{2}\right) d\mathbf{y}. \end{aligned}$$

Formal derivation of RTE for  $W_{11}^\varepsilon$  is classic; see [11]. We observe, however, that the equation for  $W_{12}^\varepsilon$  and  $W_{22}^\varepsilon$  are not standard because of the shift  $\phi$ . In fact, correlation of wave fields in different random media has been considered in [1], but they do not cover the case when one medium is a local shift of the other as described in (7).

Now we adapt the formal derivations in [11, 1] to derive RTE for the cross-correlations  $W_{12}^\varepsilon$  and  $W_{22}^\varepsilon$ . In the process of the derivation and as in the references, various assumptions will be made whose rigorous justifications are known to be difficult. To form a closed equation for  $W_{12}^\varepsilon$ , we rewrite the equation it satisfies as follows using the Fourier transform of  $V$ ,

$$i\mathbf{k} \cdot \nabla W_{12}^\varepsilon(\mathbf{x}, \mathbf{k}) = \frac{1}{\sqrt{\varepsilon}} \iint \frac{e^{i\mathbf{k} \cdot \mathbf{y}} d\mathbf{y} d\mathbf{p}}{(2\pi)^d} \hat{V}(\mathbf{p}) \left[ e^{-i\mathbf{p} \cdot (\frac{\mathbf{x}}{\varepsilon} + \frac{\mathbf{y}}{2} + \phi(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2}))} - e^{-i\mathbf{p} \cdot (\frac{\mathbf{x}}{\varepsilon} - \frac{\mathbf{y}}{2})} \right] u_1^\varepsilon(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2}) \overline{u_2^\varepsilon}(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2}).$$

We replace the function  $\phi(\mathbf{x} + \varepsilon\mathbf{y}/2)$  in the phase function by  $\phi(\mathbf{x})$  and neglect the error term of order  $\varepsilon$ . It follows that

$$\begin{aligned} i\mathbf{k} \cdot \nabla W_{12}^\varepsilon(\mathbf{x}, \mathbf{k}) &\approx \frac{1}{\sqrt{\varepsilon}} \iint \frac{e^{i\mathbf{k} \cdot \mathbf{y}} d\mathbf{y} d\mathbf{p}}{(2\pi)^d} \hat{V}(\mathbf{p}) \left[ e^{-i\mathbf{p} \cdot (\frac{\mathbf{x}}{\varepsilon} + \frac{\mathbf{y}}{2} + \phi(\mathbf{x}))} - e^{-i\mathbf{p} \cdot (\frac{\mathbf{x}}{\varepsilon} - \frac{\mathbf{y}}{2})} \right] u_1^\varepsilon(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2}) \overline{u_2^\varepsilon}(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2}) \\ &= \frac{1}{\sqrt{\varepsilon}} \int e^{-i\mathbf{p} \cdot \frac{\mathbf{x}}{\varepsilon}} \hat{V}(\mathbf{p}) \left[ e^{-i\mathbf{p} \cdot \phi(\mathbf{x})} W_{12}^\varepsilon(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2}) - W_{12}^\varepsilon(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2}) \right] d\mathbf{p}, \end{aligned}$$

where in the second equality we have used the definition (9). By the same argument we also have

$$i\mathbf{k} \cdot \nabla W_{22}^\varepsilon(\mathbf{x}, \mathbf{k}) = \frac{1}{\sqrt{\varepsilon}} \int e^{-i\mathbf{p} \cdot (\frac{\mathbf{x}}{\varepsilon} + \phi(\mathbf{x}))} \hat{V}(\mathbf{p}) \left[ W_{22}^\varepsilon(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2}) - W_{22}^\varepsilon(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2}) \right] d\mathbf{p}.$$

### 3.1 Radiative transfer limit by multiscale expansion

In this subsection we consider the limit of  $W_{12}^\varepsilon$  and  $W_{22}^\varepsilon$  as  $\varepsilon$  goes to zero using a formal multiscale expansion argument. The equations derived earlier can be written as

$$\mathbf{k} \cdot \nabla W_{jl}^\varepsilon(\mathbf{x}, \mathbf{k}) + \frac{1}{\sqrt{\varepsilon}} \mathcal{P}_{jl} W_{jl}^\varepsilon(\mathbf{x}, \mathbf{k}) = 0. \quad (12)$$

Here  $jl$  takes the value 12 or 22. The operator  $\mathcal{P}_{jl}$  is defined accordingly by

$$\begin{aligned} \mathcal{P}_{12} W_{12}^\varepsilon &= i \int e^{-i\mathbf{p} \cdot \frac{\mathbf{x}}{\varepsilon}} \hat{V}(\mathbf{p}) \left[ e^{-i\mathbf{p} \cdot \phi(\mathbf{x})} W_{12}^\varepsilon(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2}) - W_{12}^\varepsilon(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2}) \right] d\mathbf{p}, \\ \mathcal{P}_{22} W_{22}^\varepsilon &= i \int e^{-i\mathbf{p} \cdot (\frac{\mathbf{x}}{\varepsilon} + \phi(\mathbf{x}))} \hat{V}(\mathbf{p}) \left[ W_{22}^\varepsilon(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2}) - W_{22}^\varepsilon(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2}) \right] d\mathbf{p}. \end{aligned}$$

Following the method of multiscale expansion, we define the fast variable  $\boldsymbol{\xi} = \mathbf{x}/\varepsilon$ , and assume that the following expansion is valid:

$$W_{jl}^\varepsilon(\mathbf{x}, \mathbf{k}) = W_{jl}^{(0)}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}) + \sqrt{\varepsilon} W_{jl}^{(1)}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}) + \varepsilon W_{jl}^{(2)}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}) + \dots \quad (13)$$

As  $\varepsilon$  goes to zero, the leading-order term  $W_{jl}^{(0)}$  dominates, and the goal is to derive a RTE for this term. To this end, we substitute the multiscale ansatz (13) and the relation

$$\nabla = \nabla_{\mathbf{x}} + \frac{1}{\varepsilon} \nabla_{\boldsymbol{\xi}}$$

into (12). Equating terms that are of equal order in  $\varepsilon$ , we find:

$$\mathcal{O}(\varepsilon^{-1}) : \quad \mathbf{k} \cdot \nabla_{\boldsymbol{\xi}} W_{jl}^{(0)} = 0, \quad (14)$$

$$\mathcal{O}(\varepsilon^{-1/2}) : \quad \mathbf{k} \cdot \nabla_{\boldsymbol{\xi}} W_{jl}^{(1)} + \mathcal{P}_{jl} W_{jl}^{(0)} = 0, \quad (15)$$

$$\mathcal{O}(1) : \quad \mathbf{k} \cdot \nabla_{\mathbf{x}} W_{jl}^{(0)} + \mathbf{k} \cdot \nabla_{\boldsymbol{\xi}} W_{jl}^{(2)} + \mathcal{P}_{jl} W_{jl}^{(1)} = 0. \quad (16)$$

The strategy of the derivation is as follows: The first equation imposes that  $W_{jl}^{(0)}$  is independent of the fast variable  $\boldsymbol{\xi}$ . The second equation can be explicitly inverted and provides a representation of  $W_{jl}^{(1)}$  in terms of  $W_{jl}^{(0)}$ . Take statistical averaging on the third equation, and observe that

$$\mathbb{E}[\mathbf{k} \cdot \nabla_{\boldsymbol{\xi}} W_{jl}^{(2)}] = 0,$$

which is true because the statistics of  $V$  is homogeneous in the fast variable  $\boldsymbol{\xi}$ ; see the statement above (3). We also note that it is natural to assume that  $W_{jl}^{(0)}$  is not random. Then the third equation reduces to

$$\mathbf{k} \cdot \nabla_{\mathbf{x}} W_{jl}^{(0)} + \mathbb{E}[\mathcal{P}_{jl} W_{jl}^{(1)}] = 0.$$

Therefore, it suffices to invert (15) and evaluate the expectation above.

**Limiting equation for  $W_{12}^\varepsilon$ .** Upon adding an absorption term  $\theta W_{12}^{(1)}$  for regularization, we rewrite (15) as

$$\mathbf{k} \cdot \nabla_{\boldsymbol{\xi}} W_{12}^{(1)} + \theta W_{12}^{(1)} + i \int e^{-i\mathbf{p} \cdot \boldsymbol{\xi}} \hat{V}(\mathbf{p}) \left[ e^{-i\mathbf{p} \cdot \boldsymbol{\phi}(\mathbf{x})} W_{12}^{(0)}(\mathbf{k} - \frac{\mathbf{p}}{2}) - W_{12}^{(0)}(\mathbf{k} + \frac{\mathbf{p}}{2}) \right] d\mathbf{p} = 0.$$

Taking Fourier transform in the fast variable  $\boldsymbol{\xi}$ , we obtain from the above equation that

$$\hat{W}_{12}^{(1)}(\mathbf{x}, \boldsymbol{\zeta}, \mathbf{k}) = \frac{\hat{V}(\boldsymbol{\zeta}) \left[ e^{-i\boldsymbol{\zeta} \cdot \boldsymbol{\phi}(\mathbf{x})} W_{12}^{(0)}(\mathbf{k} - \frac{\boldsymbol{\zeta}}{2}) - W_{12}^{(0)}(\mathbf{k} + \frac{\boldsymbol{\zeta}}{2}) \right]}{\mathbf{k} \cdot \boldsymbol{\zeta} + i\theta}. \quad (17)$$

To estimate the expectation of  $\mathcal{P}_{12} W_{12}^{(1)}$ , we rewrite this integral using the Fourier transform of  $W_{12}^{(1)}$  and get

$$\mathcal{P}_{12} W_{12}^{(1)}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}) = i \iint e^{-i(\mathbf{p}+\mathbf{q}) \cdot \boldsymbol{\xi}} \hat{V}(\mathbf{p}) \left[ e^{-i\mathbf{p} \cdot \boldsymbol{\phi}(\mathbf{x})} \hat{W}_{12}^{(1)}(\mathbf{x}, \mathbf{q}, \mathbf{k} - \frac{\mathbf{p}}{2}) - \hat{W}_{12}^{(1)}(\mathbf{x}, \mathbf{q}, \mathbf{k} + \frac{\mathbf{p}}{2}) \right] d\mathbf{q} d\mathbf{p}.$$

Using (17), we can represent this integral in terms of  $W_{12}^{(0)}$  as follows:

$$\begin{aligned} \mathcal{P}_{12} W_{12}^{(1)}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}) = & i \iint e^{-i(\mathbf{p}+\mathbf{q}) \cdot \boldsymbol{\xi}} \hat{V}(\mathbf{p}) \hat{V}(\mathbf{q}) \\ & \times \left\{ \frac{e^{-i\mathbf{p} \cdot \boldsymbol{\phi}(\mathbf{x})} \left[ e^{-i\mathbf{q} \cdot \boldsymbol{\phi}(\mathbf{x})} W_{12}^{(0)}(\mathbf{k} - \frac{\mathbf{p}}{2} - \frac{\mathbf{q}}{2}) - W_{12}^{(0)}(\mathbf{k} - \frac{\mathbf{p}}{2} + \frac{\mathbf{q}}{2}) \right]}{(\mathbf{k} - \frac{\mathbf{p}}{2}) \cdot \mathbf{q} + i\theta} \right. \\ & \left. - \frac{e^{-i\mathbf{q} \cdot \boldsymbol{\phi}(\mathbf{x})} W_{12}^{(0)}(\mathbf{k} + \frac{\mathbf{p}}{2} - \frac{\mathbf{q}}{2}) - W_{12}^{(0)}(\mathbf{k} + \frac{\mathbf{p}}{2} + \frac{\mathbf{q}}{2})}{(\mathbf{k} + \frac{\mathbf{p}}{2}) \cdot \mathbf{q} + i\theta} \right\} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

Taking expectation, using the definition (5) and the fact that  $\hat{R}(\mathbf{p}) = \hat{R}(-\mathbf{p})$ , we obtain that the expectation  $\mathcal{P}_{12}W_{12}^{(1)}$  can be evaluated as

$$\begin{aligned}
& \mathbb{E}[\mathcal{P}_{12}W_{12}^{(1)}] \\
&= i \int \hat{R}(\mathbf{p}) \left\{ \frac{W_{12}^{(0)}(\mathbf{k}) - e^{-i\mathbf{p}\cdot\phi(\mathbf{x})}W_{12}^{(0)}(\mathbf{k}-\mathbf{p})}{-(\mathbf{k}-\frac{\mathbf{p}}{2})\cdot\mathbf{p} + i\theta} - \frac{e^{i\mathbf{p}\cdot\phi(\mathbf{x})}W_{12}^{(0)}(\mathbf{k}+\mathbf{p}) - W_{12}^{(0)}(\mathbf{k})}{-(\mathbf{k}+\frac{\mathbf{p}}{2})\cdot\mathbf{p} + i\theta} \right\} d\mathbf{p} \\
&= i \int \hat{R}(\mathbf{k}-\mathbf{p}) \frac{W_{12}^{(0)}(\mathbf{k}) - e^{-i(\mathbf{k}-\mathbf{p})\cdot\phi(\mathbf{x})}W_{12}^{(0)}(\mathbf{p})}{-\frac{1}{2}(|\mathbf{k}|^2 - |\mathbf{p}|^2) + i\theta} + \hat{R}(\mathbf{p}-\mathbf{k}) \frac{e^{i(\mathbf{p}-\mathbf{k})\cdot\phi(\mathbf{x})}W_{12}^{(0)}(\mathbf{p}) - W_{12}^{(0)}(\mathbf{k})}{-\frac{1}{2}(|\mathbf{k}|^2 - |\mathbf{p}|^2) - i\theta} d\mathbf{p} \\
&= i \int \hat{R}(\mathbf{p}-\mathbf{k}) [W_{12}^{(0)}(\mathbf{k}) - W_{12}^{(0)}(\mathbf{p})e^{i(\mathbf{p}-\mathbf{k})\cdot\phi(\mathbf{x})}] \frac{-2i\theta}{\frac{1}{4}(|\mathbf{k}|^2 - |\mathbf{p}|^2)^2 + \theta^2} d\mathbf{p} \\
&\xrightarrow{\theta \rightarrow 0} 4\pi \int \hat{R}(\mathbf{p}-\mathbf{k}) [W_{12}^{(0)}(\mathbf{k}) - W_{12}^{(0)}(\mathbf{p})e^{i(\mathbf{p}-\mathbf{k})\cdot\phi(\mathbf{x})}] \delta(|\mathbf{k}|^2 - |\mathbf{p}|^2) d\mathbf{p}.
\end{aligned}$$

In the first equality, we have used the change of variables  $\mathbf{k}-\mathbf{p} \mapsto \mathbf{p}$  and  $\mathbf{k}+\mathbf{p} \mapsto \mathbf{p}$  for the two integrands respectively. In the last equality, we have used the fact that

$$\frac{\theta}{x^2 + \theta^2} \xrightarrow{\theta \rightarrow 0} \pi \delta(x)$$

as a distribution of the one-dimensional variable  $x$ . Finally, we obtain the following RTE for  $W_{12}^{(0)}$ :

$$\mathbf{k} \cdot \nabla W_{12}^{(0)}(\mathbf{x}, \mathbf{k}) + 4\pi \int \hat{R}(\mathbf{p}-\mathbf{k}) [W_{12}^{(0)}(\mathbf{k}) - W_{12}^{(0)}(\mathbf{p})e^{i(\mathbf{p}-\mathbf{k})\cdot\phi(\mathbf{x})}] \delta(|\mathbf{k}|^2 - |\mathbf{p}|^2) d\mathbf{p} = 0. \quad (18)$$

**Limiting equation for  $W_{22}^\varepsilon$ .** The above procedure can be applied to the equation satisfied by  $W_{22}^\varepsilon$  as well. Upon adding a regularizing term and using Fourier transform, we solve equation (15) by

$$\hat{W}_{22}^{(1)}(\mathbf{x}, \boldsymbol{\zeta}, \mathbf{k}) = \frac{\hat{V}(\boldsymbol{\zeta})e^{-i\boldsymbol{\zeta}\cdot\phi(\mathbf{x})} \left[ W_{22}^{(0)}(\mathbf{k} - \frac{\boldsymbol{\zeta}}{2}) - W_{22}^{(0)}(\mathbf{k} + \frac{\boldsymbol{\zeta}}{2}) \right]}{\mathbf{k} \cdot \boldsymbol{\zeta} + i\theta}. \quad (19)$$

Using this solution and the Fourier transform representation, we find that  $\mathcal{P}_{22}W_{22}^{(1)}$  has the form

$$\begin{aligned}
\mathcal{P}_{22}W_{22}^{(1)}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k}) &= i \iint e^{-i(\mathbf{p}+\mathbf{q})\cdot(\boldsymbol{\xi}+\phi(\mathbf{x}))} \hat{V}(\mathbf{p}) \hat{V}(\mathbf{q}) \left\{ \frac{W_{22}^{(0)}(\mathbf{k} - \frac{\mathbf{p}}{2} - \frac{\mathbf{q}}{2}) - W_{22}^{(0)}(\mathbf{k} - \frac{\mathbf{p}}{2} + \frac{\mathbf{q}}{2})}{(\mathbf{k} - \frac{\mathbf{p}}{2})\cdot\mathbf{q} + i\theta} \right. \\
&\quad \left. - \frac{W_{22}^{(0)}(\mathbf{k} + \frac{\mathbf{p}}{2} - \frac{\mathbf{q}}{2}) - W_{22}^{(0)}(\mathbf{k} + \frac{\mathbf{p}}{2} + \frac{\mathbf{q}}{2})}{(\mathbf{k} + \frac{\mathbf{p}}{2})\cdot\mathbf{q} + i\theta} \right\} d\mathbf{p} d\mathbf{q}.
\end{aligned}$$

This expression is much simpler compared with that of  $\mathcal{P}_{12}W_{12}^{(1)}$  because the phase modification due to  $\phi$  is uniform for the  $W_{22}^{(0)}$  above. Taking expectation and using (5) we find that this modification has no effect on the expectation. In fact, we have

$$\mathbb{E}[\mathcal{P}_{22}W_{22}^{(1)}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{k})] \rightarrow 4\pi \int \hat{R}(\mathbf{p}-\mathbf{k}) [W_{22}^{(0)}(\mathbf{k}) - W_{22}^{(0)}(\mathbf{p})] \delta(|\mathbf{k}|^2 - |\mathbf{p}|^2) d\mathbf{p}.$$

Finally, we obtain the RTE for  $W_{22}^{(0)}$ :

$$\mathbf{k} \cdot \nabla W_{22}^{(0)}(\mathbf{x}, \mathbf{k}) + 4\pi \int \hat{R}(\mathbf{p} - \mathbf{k}) [W_{22}^{(0)}(\mathbf{k}) - W_{22}^{(0)}(\mathbf{p})] \delta(|\mathbf{k}|^2 - |\mathbf{p}|^2) d\mathbf{p} = 0. \quad (20)$$

We note that this is the same as the classic RTE limit for  $W_{11}^{(0)}$ .

**Summary of the results.** Now we summarize the above results and discuss the boundary conditions when the problem is posed on a bounded domain as often encountered in practice. Noticing that

$$\delta(|\mathbf{k}|^2 - |\mathbf{p}|^2) = \frac{1}{2|\mathbf{k}|} \delta(|\mathbf{k}| - |\mathbf{p}|),$$

we define the scattering kernel

$$\sigma(\hat{\mathbf{p}}, \hat{\mathbf{k}}; |\mathbf{k}|) = 2\pi |\mathbf{k}|^{d-3} \hat{R}(\mathbf{p} - \mathbf{k}), \quad (21)$$

and the absorption coefficient

$$\Sigma(\mathbf{k}) = \frac{4\pi}{|\mathbf{k}|} \int \delta(|\mathbf{k}|^2 - |\mathbf{p}|^2) \hat{R}(\mathbf{p} - \mathbf{k}) d\mathbf{p} = \int_{S^{d-1}} \sigma(\hat{\mathbf{p}}, \hat{\mathbf{k}}; |\mathbf{k}|) d\hat{\mathbf{p}}. \quad (22)$$

Here and in the sequel, the vector  $\hat{\mathbf{p}}$  is defined to be the direction  $\mathbf{p}/|\mathbf{p}|$  for non-zero  $\mathbf{p}$ . Abusing notations we use  $W_{jl}$  for the leading-order terms  $W_{jl}^{(0)}$ . With these notations, the equations (18) and (20) become

$$\hat{\mathbf{k}} \cdot \nabla W_{12} + \Sigma W_{12} = \int_{S^{d-1}} \sigma(\hat{\mathbf{p}}, \hat{\mathbf{k}}; |\mathbf{k}|) e^{i|\mathbf{k}|(\hat{\mathbf{p}} - \hat{\mathbf{k}}) \cdot \boldsymbol{\phi}(\mathbf{x})} W_{12}(\mathbf{x}, |\mathbf{k}| \hat{\mathbf{p}}) d\hat{\mathbf{p}}, \quad (23)$$

$$\hat{\mathbf{k}} \cdot \nabla W_{jj} + \Sigma W_{jj} = \int_{S^{d-1}} \sigma(\hat{\mathbf{p}}, \hat{\mathbf{k}}; |\mathbf{k}|) W_{jj}(\mathbf{x}, |\mathbf{k}| \hat{\mathbf{p}}) d\hat{\mathbf{p}}, \quad j = 1, 2. \quad (24)$$

On a bounded domain  $X \subset \mathbb{R}^d$ , we need to equip the RTEs with proper boundary conditions. We consider the monokinetic case when  $|\mathbf{k}|$  is fixed, so the energy density is specified by the spatial variable  $\mathbf{x}$  and the direction  $\hat{\mathbf{k}}$ . The RTEs are therefore posed on the phase space  $X \times S^{d-1}$ , with  $|\mathbf{k}|$  as a fixed parameter. We define the incoming boundary  $\Gamma_-$  and the outgoing boundary  $\Gamma_+$  as

$$\Gamma_{\pm} := \{(\mathbf{x}, \hat{\mathbf{k}}) \mid \mathbf{x} \in \partial X, \pm \hat{\mathbf{k}} \cdot \boldsymbol{\nu}(\mathbf{x}) > 0\}, \quad (25)$$

where  $\boldsymbol{\nu}(\mathbf{x})$  is the outer-pointing normal at the point  $\mathbf{x}$  on the boundary  $\partial X$ . When a bounded domain is considered, the RTEs (23) and (24) for the Wigner transforms  $W_{jl}$ 's should be understood as for  $(\mathbf{x}, \hat{\mathbf{k}}) \in X \times S^{d-1}$  with the boundary condition

$$W_{jl}(\mathbf{x}, \hat{\mathbf{k}}) = p(\mathbf{x}, \hat{\mathbf{k}}), \quad (\mathbf{x}, \hat{\mathbf{k}}) \in \Gamma_-, \quad (26)$$

where  $p$  models the light intensity incoming at point  $\mathbf{x} \in \partial X$  in the direction  $\hat{\mathbf{k}}$ . In the case of an incident laser beam the support of  $p$  is spatially limited to  $\mathbf{x} \in \partial X_i$  where  $\partial X_i$  is the part of  $\partial X$  where the laser beam is applied.

**The case of large shifts.** When the shift  $\boldsymbol{\phi}$  is much larger than the wavelength, i.e.,

$$|\mathbf{k}| |\boldsymbol{\phi}| \gg 1, \quad (27)$$



the RTE for  $W_{12}$  can be simplified. Indeed, since the integral is taken over the sphere which has non-vanishing Gaussian curvature, we can apply [12, Theorem 1.2.1] which may be viewed as an analog of the Riemann-Lebesgue lemma, and conclude that the integral on the right-hand side of (18) is of order  $(|\mathbf{k}||\phi|)^{-(d-1)/2}$  and hence approaches zero. Consequently, (18) should be modified as follows

$$\begin{aligned}\hat{\mathbf{k}} \cdot \nabla W_{12} + \Sigma W_{12} &= 0, & (\mathbf{x}, \hat{\mathbf{k}}) &\in X_s \times S^{d-2}, \\ \hat{\mathbf{k}} \cdot \nabla W_{12} + \Sigma W_{12} &= \int_{S^{d-1}} \sigma(\hat{\mathbf{p}}, \hat{\mathbf{k}}; |\mathbf{k}|) W_{12}(\mathbf{x}, |\mathbf{k}| \hat{\mathbf{p}}) d\hat{\mathbf{p}}, & (\mathbf{x}, \hat{\mathbf{k}}) &\in X_s^c \times S^{d-2}.\end{aligned}\quad (28)$$

Here  $X_s$  is the support of  $\phi$ , *i.e.*, the region in which the scatterers are shifted, and  $X_s^c$  is the complement of  $X_s$  in  $X$ .

### 3.2 Speckle pattern correlations in the RTE regime

As we have seen, the Wigner transform  $W[u, u]$  of a wave field  $u$  has the interpretation of direction-resolved energy density. Hence RTE for  $W[u, u]$  is a very good model for light propagation. In this section, we relate the correlation of optical speckle patterns to integrals of Wigner transforms.

In the opto-elastography experiment, emitted light intensity is measured at a part  $\partial X_m$  of the domain boundary  $\partial X$  and the data are  $\{|u_j^e(\mathbf{x})|^2 \mid \mathbf{x} \in \partial X_m\}$ ,  $j = 1, 2$ . The correlation of two speckle patterns  $u_1^e$  and  $u_2^e$  is defined by

$$C_{12} = \frac{\langle (|u_1^e|^2 - \langle |u_1^e|^2 \rangle)(|u_2^e|^2 - \langle |u_2^e|^2 \rangle) \rangle}{\sqrt{\langle (|u_1^e|^2 - \langle |u_1^e|^2 \rangle)^2 \rangle} \sqrt{\langle (|u_2^e|^2 - \langle |u_2^e|^2 \rangle)^2 \rangle}}, \quad (29)$$

where  $\langle A \rangle$  denotes the spatial average over the boundary  $\partial X_m$ , that is

$$\langle A \rangle = \frac{1}{|\partial X_m|} \int_{\partial X_m} A(\mathbf{x}) d\mathbf{x}. \quad (30)$$

Note that in the above equation, the notation  $d\mathbf{x}$  means the induced Lebesgue measure on the boundary  $\partial X_m$ , and  $|\partial X_m|$  is the area of the boundary.

Assume that the complex amplitudes  $(u_1^e, u_2^e)$ , as a  $\mathbb{C}^2$ -valued random process, satisfy the circular symmetric Gaussian distribution, and assume also that the spatial average can be thought as ensemble averages (taking expectations), we have

$$\langle |u_j^e|^2 |u_l^e|^2 \rangle = \langle u_j^e \overline{u_l^e} \rangle \langle \overline{u_j^e} u_l^e \rangle + \langle |u_j^e|^2 \rangle \langle |u_l^e|^2 \rangle, \quad j, l = 1, 2. \quad (31)$$

Using this equality, we find that the numerator of (29) is

$$|\langle u_1^e \overline{u_2^e} \rangle|^2 = \frac{1}{|\partial X_m|^2} \left| \int_{\partial X_m} u_1^e(\mathbf{x}) \overline{u_2^e}(\mathbf{x}) d\mathbf{x} \right|^2 = \frac{1}{|\partial X_m|^2} \left| \int_{\Gamma_{m,+}} W_{12}(\mathbf{x}, \hat{\mathbf{k}}) d\hat{\mathbf{k}} d\mathbf{x} \right|^2,$$

where  $\Gamma_{m,+} := \{(\mathbf{x}, \hat{\mathbf{k}}) \mid \mathbf{x} \in \partial X_m, \hat{\mathbf{k}} \cdot \boldsymbol{\nu}(\mathbf{x}) > 0\}$ . In the second equality above, we have used the fact that  $u_1^e \overline{u_2^e}$  is the integral over  $\hat{\mathbf{k}}$  of  $W_{12}(\mathbf{x}, \hat{\mathbf{k}})$ , as seen in (10). Since the product  $u_1^e \overline{u_2^e}$  only accounts for the outgoing light, we take the integral of  $W_{12}$  over outgoing directions. Similarly, the denominator in (29) is given by

$$\langle |u_1^e|^2 \rangle \langle |u_2^e|^2 \rangle = \frac{1}{|\partial X_m|^2} \int_{\Gamma_{m,+}} W_{11}(\mathbf{x}, \hat{\mathbf{k}}) d\hat{\mathbf{k}} d\mathbf{x} \int_{\Gamma_{m,+}} W_{22}(\mathbf{x}, \hat{\mathbf{k}}) d\hat{\mathbf{k}} d\mathbf{x}.$$

Combining these calculations above, we find that in the RTE regime, the correlation of two speckle patterns is:

$$C_{12} = \frac{\left| \int_{\Gamma_{m,+}} W_{12}(\mathbf{x}, \hat{\mathbf{k}}) d\hat{\mathbf{k}} d\mathbf{x} \right|^2}{\int_{\Gamma_{m,+}} W_{11}(\mathbf{x}, \hat{\mathbf{k}}) d\hat{\mathbf{k}} d\mathbf{x} \int_{\Gamma_{m,+}} W_{22}(\mathbf{x}, \hat{\mathbf{k}}) d\hat{\mathbf{k}} d\mathbf{x}}. \quad (32)$$

We remark that modeling of the outgoing light depends on the measurement set-up. In the above model, outgoing light in any directions are completely captured. If light is collected in the far field by a lens, then we should only take the integral in  $\hat{\mathbf{k}}$  in a cone that corresponds to the aperture of the lens. Such a situation does not affect qualitatively the results in the sequel.

## 4 The diffusion limit for the radiative transfer equations

The RTEs derived in the previous section are posed on the phase space  $X \times S^{d-1}$ . This causes difficulties for instance for numerical simulations. In this section, we consider their diffusion approximations which are much easier to deal with for the purpose of simulations.

For the sake of simplicity, we consider the case when the scattering kernel  $\sigma$ , as defined in (21), takes the form

$$\sigma(\hat{\mathbf{p}}, \hat{\mathbf{k}}; |\mathbf{k}|) = \sigma(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}; |\mathbf{k}|).$$

In this case, the total absorption section  $\Sigma$  defined in (22) depends only on the parameter  $|\mathbf{k}|$ . In the sequel of this paper, we will use the normalized scattering kernel

$$f(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}; |\mathbf{k}|) = \frac{1}{\Sigma(|\mathbf{k}|)} \sigma(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}; |\mathbf{k}|). \quad (33)$$

Very often and as we have already done so, the dependence on  $|\mathbf{k}|$  is not written explicitly. The diffusion approximation is valid in the regime where the mean free path  $\eta = 1/\Sigma$  is much smaller than the size of the domain. Here we assume  $\eta \ll 1$  and the domain is of order one<sup>1</sup>.

In the following two subsections, we adapt the classical derivation of diffusion limits of RTE [8, 3, 7] to the case considered in this paper. The point here is that we have to manage the shift field  $\phi$ . Though rigorous derivation of diffusion limit is possible, we do not pursue it here and adopt the formal multiscale expansion argument only. We consider two interesting situations. In the first one, the amplitude of the shift  $\phi$  is very small so that  $|\mathbf{k}||\phi|$  is of order  $\eta$ . In the second one, the amplitude of the shift  $\phi$  is very large so that  $|\mathbf{k}||\phi|$  is much larger than one. As seen in the last section, this leads to the RTE (28).

### 4.1 The case of small shift

In this section, the amplitude of the shift  $\phi$  is assumed to be small so that  $|\mathbf{k}|\phi(\mathbf{x}) = \eta\psi(\mathbf{x})$ . The diffusion limit for  $W_{11}$  and  $W_{22}$  are classic [7, Section XXI.5.4]; hence we concentrate on

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<sup>1</sup>Alternatively, when  $\eta$  is bounded but not necessarily small, diffusion limit is still valid when we consider the problem on a very large scale and rescale the spatial variable.

that for  $W_{12}$  in (23). Dividing on both sides of this equation by  $\Sigma$ , we get

$$\eta \hat{\mathbf{k}} \cdot \nabla W_{12}(\mathbf{x}, \hat{\mathbf{k}}) = \int_{S^{d-1}} f(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) \left[ e^{i\eta(\hat{\mathbf{p}} - \hat{\mathbf{k}}) \cdot \psi(\mathbf{x})} W_{12}(\mathbf{x}, \hat{\mathbf{p}}) - W_{12}(\mathbf{x}, \hat{\mathbf{k}}) \right] d\hat{\mathbf{p}}.$$

We follow the idea used in [1] and define a new function which takes account the phase shift

$$\widetilde{W}_{12}(\mathbf{x}, \hat{\mathbf{k}}) = e^{i\eta \hat{\mathbf{k}} \cdot \psi(\mathbf{x})} W_{12}(\mathbf{x}, \hat{\mathbf{k}}).$$

Then it is easy to verify that  $\widetilde{W}_{12}$  satisfies

$$\eta \hat{\mathbf{k}} \cdot \nabla \widetilde{W}_{12} - i\eta^2 \Sigma_a \widetilde{W}_{12} = \int_{S^{d-1}} f(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) \left[ \widetilde{W}_{12}(\mathbf{x}, \hat{\mathbf{p}}) - \widetilde{W}_{12}(\mathbf{x}, \hat{\mathbf{k}}) \right] d\hat{\mathbf{p}}. \quad (34)$$

Here, the intrinsic “complex absorption” coefficient is given by  $\Sigma_a = \text{Tr}(\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} \nabla \psi(\mathbf{x}))$ , where  $\hat{\mathbf{k}} \otimes \hat{\mathbf{k}}$  is the projection matrix  $\hat{\mathbf{k}} \hat{\mathbf{k}}^t$  and  $\text{Tr}$  means taking the trace.

Since  $\widetilde{W}_{12}$  approximates  $W_{12}$  as  $\eta$  goes to zero, it suffices to consider the limit of  $\widetilde{W}_{12}$ . Abusing notations, we still denote this function by  $W_{12}$ . To start the formal derivation, we substitute the ansatz

$$W_{12} = W_{12}^{(0)} + \eta W_{12}^{(1)} + \eta^2 W_{12}^{(2)} + \dots \quad (35)$$

into the equation (34). Equating the terms that are of equal order in  $\eta$  we find

$$\mathcal{O}(1): \quad 0 = \int_{S^{d-1}} f(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) \left[ W_{12}^{(0)}(\hat{\mathbf{p}}) - W_{12}^{(0)}(\hat{\mathbf{k}}) \right] d\hat{\mathbf{p}}, \quad (36)$$

$$\mathcal{O}(\eta): \quad \hat{\mathbf{k}} \cdot \nabla W_{12}^{(0)} = \int_{S^{d-1}} f(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) \left[ W_{12}^{(1)}(\hat{\mathbf{p}}) - W_{12}^{(1)}(\hat{\mathbf{k}}) \right] d\hat{\mathbf{p}}, \quad (37)$$

$$\mathcal{O}(\eta^2): \quad \left( \hat{\mathbf{k}} \cdot \nabla W_{12}^{(1)} - i\Sigma_a W_{12}^{(0)} \right) = \int_{S^{d-1}} f(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) \left[ W_{12}^{(2)}(\hat{\mathbf{p}}) - W_{12}^{(2)}(\hat{\mathbf{k}}) \right] d\hat{\mathbf{p}}. \quad (38)$$

To study these equations, we define the integral operator

$$Kh(\hat{\mathbf{k}}) := \int_{S^{d-1}} f(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) h(\hat{\mathbf{p}}) d\hat{\mathbf{p}}. \quad (39)$$

Using this definition, Eq. (36) can be recast as  $(K - I)W^{(0)} = 0$ . We verify that  $\int_{S^{d-1}} f d\hat{\mathbf{p}} = 1$ ; we assume also that  $f$  is uniformly bounded from up and below by positive numbers. In this case, the constant function is known to be the only eigenvector of  $K$  corresponding to the eigenvalue one [7]. Furthermore, the integral equation  $(K - I)h = v$  is solvable only if  $\int_{S^{d-1}} v d\hat{\mathbf{p}} = 0$  (Fredholm alternative).

Using these results, Eq. (36) shows that  $W_{12}^{(0)}$  does not depend on the direction variable  $\hat{\mathbf{k}}$ , so  $W_{12}^{(0)} = W_{12}^{(0)}(\mathbf{x})$ . Eq. (37) relates  $W_{12}^{(1)}$  to  $W_{12}^{(0)}$ . In fact, we need to solve the integral equation

$$(K - I)W_{12}^{(1)} = \hat{\mathbf{k}} \cdot \nabla W_{12}^{(0)}(\mathbf{x}).$$

Let  $h_j(\hat{\mathbf{k}})$  be the unique solution (whose integral over  $S^{d-1}$  is zero) to the equation

$$(K - I)h_j(\hat{\mathbf{k}}) = \mathbf{e}_j \cdot \hat{\mathbf{k}}, \quad (40)$$

where  $\mathbf{e}_j$  is the  $j$ -th unit vector in the orthonormal basis of  $\mathbb{R}^d$ . Indeed, this is possible because  $\mathbf{e}_j \cdot \hat{\mathbf{k}}$  integrates to zero on the sphere  $S^{d-1}$ . Let  $\mathbf{h}(\hat{\mathbf{k}})$  be the vector field  $(h_1, h_2, h_3)^t$ . It follows that  $W_{12}^{(1)}(\mathbf{x}, \hat{\mathbf{k}}) = \mathbf{h}(\hat{\mathbf{k}}) \cdot \nabla W_{12}^{(0)}(\mathbf{x})$ .

Substituting this representation of  $W_{12}^{(1)}$  into Eq. (38) and integrating the equation over  $S^{d-1}$ , we find that the right-hand side vanishes due to the symmetry of  $f$  and we get

$$\int_{S^{d-1}} \hat{\mathbf{k}} \cdot \nabla \left[ \mathbf{h}(\hat{\mathbf{k}}) \cdot \nabla W^{(0)}(\mathbf{x}) \right] d\hat{\mathbf{k}} - i \int_{S^{d-1}} \text{Tr}(\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} \nabla \psi(\mathbf{x})) W^{(0)}(\mathbf{x}) d\hat{\mathbf{k}} = 0.$$

Carrying out these integrals, we find

$$-D : \nabla \nabla W^{(0)}(\mathbf{x}) - \frac{\varpi_d i}{d} (\nabla \cdot \psi(\mathbf{x})) W^{(0)}(\mathbf{x}) = 0. \quad (41)$$

Here, the symbol  $:$  denotes the Frobenius inner product of two matrices,  $d$  is the space dimension, and  $\varpi_d$  is the area of the sphere  $S^{d-1}$ . The diffusion matrix  $D$  is given by

$$D := - \int_{S^{d-1}} \mathbf{h}(\hat{\mathbf{k}}) \otimes \hat{\mathbf{k}} d\hat{\mathbf{k}}. \quad (42)$$

To complete the derivation, it suffices to find  $h_j(\hat{\mathbf{k}})$  that solves (40). Define

$$h_j(\hat{\mathbf{k}}) = - \frac{\hat{\mathbf{k}} \cdot \mathbf{e}_j}{1 - g(|\mathbf{k}|)}, \quad (43)$$

with

$$g(|\mathbf{k}|) = \int_{S^{d-1}} f(\hat{\mathbf{p}} \cdot \mathbf{e}; |\mathbf{k}|) \hat{\mathbf{p}} \cdot \mathbf{e} d\hat{\mathbf{p}} = C_d \int_{-1}^1 f(\mu; |\mathbf{k}|) \mu d\mu, \quad (44)$$

where  $\mathbf{e}$  is a unit vector in  $S^{d-1}$  and  $C_d$  is a constant depending only on the dimension. We note that  $g$  is independent of the choice of  $\mathbf{e}$  and that  $g < 1$ . Let  $Q_j$  be an orthonormal matrix so that  $Q_j \hat{\mathbf{k}} = \mathbf{e}_j$ . We check that

$$\begin{aligned} (K - I)(-\hat{\mathbf{k}} \cdot \mathbf{e}_j) &= \hat{\mathbf{k}} \cdot \mathbf{e}_j - \int_{S^{d-1}} f(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) \hat{\mathbf{p}} \cdot \mathbf{e}_j d\hat{\mathbf{p}} \\ &= \hat{\mathbf{k}} \cdot \mathbf{e}_j - \int_{S^{d-1}} f(Q_j \hat{\mathbf{p}} \cdot \mathbf{e}_j) Q_j \hat{\mathbf{p}} \cdot Q_j \mathbf{e}_j d\hat{\mathbf{p}} = \hat{\mathbf{k}} \cdot \mathbf{e}_j - \int_{S^{d-1}} f(\hat{\mathbf{p}} \cdot \mathbf{e}_j) \hat{\mathbf{p}} \cdot Q_j \mathbf{e}_j d\hat{\mathbf{p}}. \end{aligned}$$

In the last equality, we changed the variable  $Q_j \hat{\mathbf{p}}$  to  $\hat{\mathbf{p}}$ . We have the decomposition

$$Q_j \mathbf{e}_j = [\mathbf{e}_j \cdot Q_j \mathbf{e}_j] \mathbf{e}_j + \mathbf{c} \mathbf{e}_{j\perp} = [Q_j \hat{\mathbf{k}} \cdot Q_j \mathbf{e}_j] \mathbf{e}_j + \mathbf{c} \mathbf{e}_{j\perp} = (\hat{\mathbf{k}} \cdot \mathbf{e}_j) \mathbf{e}_j + \mathbf{c} \mathbf{e}_{j\perp},$$

where  $\mathbf{c} \mathbf{e}_{j\perp}$  is perpendicular to  $\mathbf{e}_j$ . By symmetry, the contribution of  $\mathbf{c} \mathbf{e}_{j\perp}$  to the spherical integral above vanishes. We can then check that

$$(K - I)(-\hat{\mathbf{k}} \cdot \mathbf{e}_j) = \hat{\mathbf{k}} \cdot \mathbf{e}_j - (\hat{\mathbf{k}} \cdot \mathbf{e}_j) \int_{S^{d-1}} f(\hat{\mathbf{p}} \cdot \mathbf{e}_j) \hat{\mathbf{p}} \cdot \mathbf{e}_j d\hat{\mathbf{p}} = (1 - g(|\mathbf{k}|)) \mathbf{e}_j \cdot \hat{\mathbf{k}}.$$

Since  $h_j$  defined in (43) clearly integrates to zero on  $S^{d-1}$ , the above calculation shows that  $h_j$  solves (40). Using this solution for  $D$  in (42), we find that  $D = \varpi_d \mathbf{I} / [d(1 - g(|\mathbf{k}|))]$  where  $\mathbf{I}$  is the identity matrix. In dimension three, the limiting diffusion equation for  $W_{12}^{(0)}$  which is denoted by  $W_{12}$  becomes

$$-\nabla \cdot \frac{4\pi}{3(1 - g(|\mathbf{k}|))} \nabla W_{12}(\mathbf{x}, |\mathbf{k}|) - i \frac{4\pi}{3} (\nabla \cdot \psi(\mathbf{x})) W_{12} = 0. \quad (45)$$

For the autocorrelation functions  $W_{11}$  and  $W_{22}$ , we take  $\psi$  above to be zero and recover the classic limiting diffusion equation

$$-\nabla \cdot \frac{4\pi}{3(1-g(|\mathbf{k}|))} \nabla W_{jj}(\mathbf{x}, |\mathbf{k}|) = 0, \quad j = 1, 2. \quad (46)$$

The above diffusion equations (45) and (46) should be equipped with proper boundary conditions. We impose that

$$W_{jl}(\mathbf{x}) = q(\mathbf{x}), \quad \mathbf{x} \in \partial X. \quad (47)$$

Here,  $q(\mathbf{x})$  models the incoming light intensity at the boundary. It can be derived from the boundary condition  $p(\mathbf{x}, \hat{\mathbf{k}})$  in (26) of the RTE. If  $p$  is independent of  $\mathbf{k}$ , then  $q = p$ . The case when  $p$  is anisotropic requires a careful boundary layer analysis as developed in [6, 3], which shows that there exists a linear operator (local in  $\mathbf{x}$ ) that maps  $p$  to  $q$ . In the case when  $f$  is constant (isotropic scattering) and  $p(\mathbf{x}, \hat{\mathbf{k}}) = \tilde{p}(\mathbf{x}, \mu)$  where  $\mu = -\hat{\mathbf{k}} \cdot \boldsymbol{\nu}(\mathbf{x})$ , this map is given by

$$q(\mathbf{x}) = \int_0^1 \tilde{p}(\mathbf{x}, \mu) H(\mu) \frac{\mu}{2} d\mu, \quad (48)$$

where  $H(\mu)$  is the so-called Chandrasekhar  $H$ -function. The evaluation of  $H$  can be found, for instance, in [3, Section 1.5]. As a result the support of  $q$  is spatially limited to the part  $\partial X_i$  of the boundary  $\partial X$  where the laser beam is applied.

## 4.2 The case of large shifts

In this section, we consider the more practical case when the shift  $\phi$  of the scatterers has large amplitude so that  $|\mathbf{k}||\phi| \gg 1$ . As we have derived in Section 3.1, the RTE for  $W_{12}$  takes the form (28). Consider the limit  $\eta = 1/\Sigma \ll 1$ , these equations can be written as

$$\begin{aligned} \eta \hat{\mathbf{k}} \cdot \nabla W_{12} + W_{12} &= 0, & (\mathbf{x}, \hat{\mathbf{k}}) &\in X_s \times S^{d-1}, \\ \eta \hat{\mathbf{k}} \cdot \nabla W_{12} + W_{12} &= \int_{S^{d-1}} f(\hat{\mathbf{k}}, \hat{\mathbf{p}}) W_{12}(\mathbf{x}, |\mathbf{k}|\hat{\mathbf{p}}) d\hat{\mathbf{p}}, & (\mathbf{x}, \hat{\mathbf{k}}) &\in X_s^c \times S^{d-1}. \end{aligned}$$

On the unshifted region  $X_s^c$ , the second line of the RTEs above takes the classic form and its diffusion limit is (46) in three dimensions. On the other hand, sending  $\eta$  to zero, we deduce from the first line of the equations above that  $W_{12}$  goes to zero in the shifted region  $X_s$  in the limit. Consequently, the diffusion limit for the cross-correlation  $W_{12}$  in the case of large shift is:

$$\begin{aligned} -\nabla \cdot \frac{4\pi}{3(1-g(|\mathbf{k}|))} \nabla W_{12}(\mathbf{x}, |\mathbf{k}|) &= 0, & \mathbf{x} &\in X_s^c, \\ W_{12} &= 0, & \mathbf{x} &\in X_s, \end{aligned} \quad (49)$$

and it is equipped with the boundary condition (47). The diffusion limits for  $W_{11}$  and  $W_{22}$  remain unchanged; that is, they are given by (46) with boundary condition (47).

## 4.3 Speckle pattern correlation in the diffusion regime

In this subsection, we revisit the formula (32) and rewrite it in terms of the functions in the diffusion limit. As before, we denote by  $\partial X_m$  the part of the boundary where measurement is taken. We naturally assume that the part  $\partial X_i$  of the boundary that is illuminated by the

incident laser beam has an empty intersection with the part  $\partial X_m$  where the outgoing light is measured. Therefore Eq. (47) implies that the leading-order term  $W_{jl}(\mathbf{x})$  in the diffusion approximation of the direction-resolved correlation function  $W_{jl}(\mathbf{x}, \hat{\mathbf{k}})$  vanishes on  $\partial X_m$ . It is then necessary to look for the first-order corrections in the expansion of  $W_{jl}(\mathbf{x}, \hat{\mathbf{k}})$ . This is discussed in detail in [10] and we follow the ideas there.

In the diffusion regime, using the expansion (35) with  $W_{12}^{(1)}$  found in section 4.1, we can write  $W_{jl}(\mathbf{x}, \hat{\mathbf{k}})$  as

$$W_{jl}(\mathbf{x}, \hat{\mathbf{k}}) = W_{jl}(\mathbf{x}) - \frac{\eta}{1 - g(|\mathbf{k}|)} \hat{\mathbf{k}} \cdot \nabla W_{jl}(\mathbf{x}) + \dots, \quad (50)$$

where  $W_{jl}(\mathbf{x})$  is the function involved in the diffusion equations (45) (46) (49). This expansion is valid only inside the domain  $X$  and not at the boundary  $\partial X$ , again because of the presence of a boundary layer which gives rise to a correction of order  $\eta$  that cancels the first-order corrective term in (50) for  $\mathbf{x} \in \partial X_m$  and  $\boldsymbol{\nu}(\mathbf{x}) \cdot \hat{\mathbf{k}} < 0$ . If, however, we carry out the calculation with this expansion formally, then we find that

$$\int_{\hat{\mathbf{k}} \cdot \boldsymbol{\nu}(\mathbf{x}) > 0} W_{jl}(\mathbf{x}, \hat{\mathbf{k}}) d\hat{\mathbf{k}} = -\frac{\eta}{1 - g(|\mathbf{k}|)} \int_{\hat{\mathbf{k}} \cdot \boldsymbol{\nu}(\mathbf{x}) > 0} \hat{\mathbf{k}} d\hat{\mathbf{k}} \cdot \nabla W_{jl}(\mathbf{x}) = -\frac{C_d \eta}{1 - g(|\mathbf{k}|)} \boldsymbol{\nu}(\mathbf{x}) \cdot \nabla W_{jl}(\mathbf{x}),$$

where  $C_d$  is a constant that depends on the dimension. The second equality follows from the decomposition  $\hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \boldsymbol{\nu}(\mathbf{x}))\boldsymbol{\nu}(\mathbf{x}) + \hat{\mathbf{k}}_{\perp}$  where  $\hat{\mathbf{k}}_{\perp}$  is perpendicular to  $\boldsymbol{\nu}$ , and the fact that the contribution of  $\hat{\mathbf{k}}_{\perp}$  averages to zero because of symmetry.

In fact the result obtained in this formal way is essentially correct up to the value of the constant  $C_d$  that may depend on the form of  $f$  and can be found numerically when  $f$  is constant in particular [10, Sec. X.A]. It follows that the correlation of speckle patterns in the diffusion regime is given by

$$C_{12} = \frac{\left| \int_{\partial X_m} \boldsymbol{\nu}(\mathbf{x}) \cdot \nabla W_{12}(\mathbf{x}) d\mathbf{x} \right|^2}{\int_{\partial X_m} \boldsymbol{\nu}(\mathbf{x}) \cdot \nabla W_{11}(\mathbf{x}) d\mathbf{x} \int_{\partial X_m} \boldsymbol{\nu}(\mathbf{x}) \cdot \nabla W_{22}(\mathbf{x}) d\mathbf{x}}. \quad (51)$$

We remark that in the above formula, the constant  $C_d \eta / (1 - g)$  is cancelled out, so the desired result does not depend on its value.

#### 4.4 Numerical simulations of the diffusion equation model

In this subsection, we show some numerical simulations which confirm that the diffusion equation model derived above for the cross correlation  $W_{12}$  captures the loss of correlation in speckle patterns. The numerical simulations are in accordance with the experimental measurements published in [4, 5].

Let the domain  $X$  be a two-dimensional square  $(-1, 1) \times (-1, 1)$ , and let a sequence of circles  $S(r_n)$  centered at  $(0, 0)$  with increasing radius  $r_n$  model the wave front of the elastic wave introduced by the ultrasound modulation. Let  $C_{12}^{(n)}$  be the correlation calculated as in (51) on the right side of the square domain  $X$  where the two random media are those when the wave fronts are at  $S(r_n)$  and  $S(r_{n-1})$  respectively. Since  $\phi$  models the shift of the scatterers of these two random media, the support of  $\phi$  is the union of the supports of the elastic waves at the two instants, and it is enclosed inside the circle  $S(r_n)$ .

To evaluate  $C_{12}^{(n)}$ , we need to calculate  $W_{11}, W_{22}$  and  $W_{12}$ . For the first two functions, we solve the diffusion equation (46) with unit diffusion coefficient on the whole domain  $X$  with boundary condition (47), which is taken as  $q = 1$  on the left side and  $q = 0$  on the three other sides of the square domain  $X$ . For  $W_{12}$ , since it vanishes on the support of  $\phi$  which has outer boundary  $S(r_n)$ , we solve the first equation of (49) on the exterior of the ball enclosed by  $S(r_n)$ ; the boundary condition is (47) on  $\partial X$  and  $W_{12} = 0$  on the inner boundary  $S(r_n)$ . These configurations of computational domains and elastic wave fronts are illustrated in Fig. 1(a).

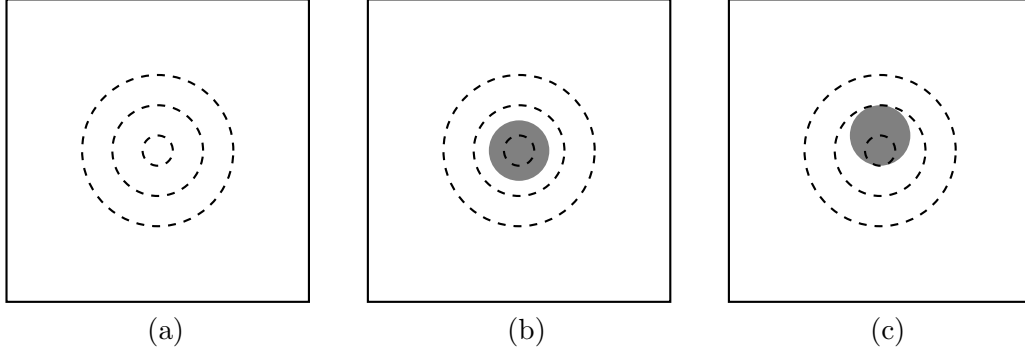


Figure 1: Computational domains and elastic wave fronts. (a) The dash lines are the wave fronts  $S(r_n)$ 's. (b) An optical absorber with radius  $r_a = 0.2$  centered at  $(0, 0)$ . (c) An optical absorber with radius  $r_a$  located at  $(0, 0.1)$ .

To demonstrate the effect of optical absorbers, we also consider the case when such an absorber with radius  $r_a = 0.2$  is located at  $(0, 0)$  and  $(0, 0.1)$  respectively, as illustrated in Fig. 1-(b)(c). They are referred to as the case with centered and non-centered absorbers respectively. In these cases, the equation for  $W_{11}$  and  $W_{22}$  are solved outside the absorber because light is completely absorbed inside. For  $W_{12}$ , the equation is solved outside the union of the absorber and the ball enclosed the circle  $S(r_n)$ .

Note that for simplicity, the diffusion coefficient  $D$  and the boundary condition  $q$  on the left boundary are chosen to have unit value. This does not affect the results of the simulation. Indeed, the diffusion constant is cancelled out in (51); further, when the boundary condition  $q$  in (48) is uniform in  $\mathbf{x}$  on the left side, its constant value will be cancelled out in (51) as well.

Finally we plot  $C_{12}^{(n)}$  as a function of  $r_n$  in Fig. 2. We see that when there is no optical absorber, the correlation of speckle patterns drops immediately when elastic wave forms. On the other hand, when optical absorber is present, the correlation starts to decay only after the elastic wave exits the absorber, that is at  $r = 0.2$  for Fig. 1(b) and at  $r = 0.1$  for Fig. 1(c). Hence the decay of correlation is sensitive to the location of the absorber. This can be exploited further for medical imaging purposes.

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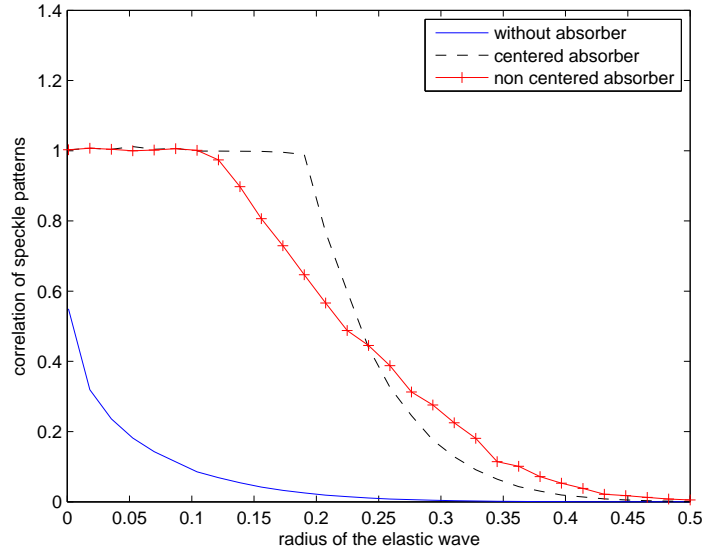


Figure 2: The correlation of consecutive speckle patterns during the propagation of a circular elastic wave.

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